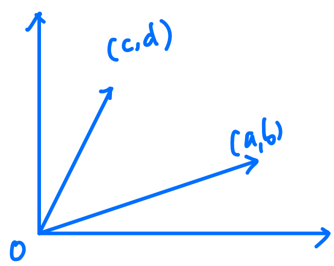
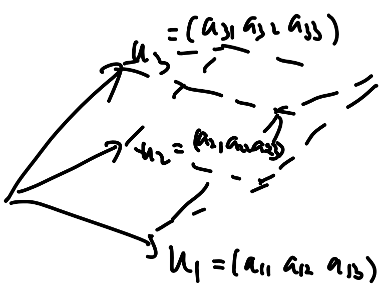
§ 行列式

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow S_{\square} = ad-bc$$





$(u_1 \times u_2)\cdot u_3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to (u_1 \times u_2)\cdot u_3$$

$$= a_{11}\cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

推广：n维空间中n个向量张成的平行多面体的有向体积

定义：方阵 $A=(a_{ij})_{n\times n}$ 的行列式记为

$$\det(A) \quad 或 \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

当 $n=1$ 时，$\det(A) := a_{11}$，当 $n\geq 2$ 时，$\det A$

$$\det(A) = \sum_{i=1}^{n} a_{1i} A_{1i} = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} M_{1i}$$

其中

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i+1,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{nn} \end{vmatrix}$$

a_{ij} 的余子式

$$= \det\left(A\begin{pmatrix} 1 \cdots i-1, i+1 \cdots n \\ 1 \cdots j-1, j+1, \cdots n \end{pmatrix}\right)$$

a_{ij}的代数余子式

$$A_{ij} := (-1)^{i+j} M_{ij}$$

$$\{1, 2, \cdots, n\} = \{i_1, \cdots, i_k\} \sqcup \{i_{k+1}, \cdots, i_n\}$$
$$= \{j_1, \cdots, j_k\} \sqcup \{j_{k+1}, \cdots, j_n\}$$

其中 $i_1 < i_2 < \cdots < i_k$, $i_{k+1} < \cdots < i_n$

$j_1 < j_2 < \cdots < j_k$, $j_{k+1} < \cdots < j_n$

$\det\left(A\begin{pmatrix} i_1\, i_2 \cdots i_k \\ j_1\, j_2 \cdots j_k \end{pmatrix}\right)$ 为 A 的 k-阶子式

$\det\left(A\begin{pmatrix} i_{k+1} \cdots i_n \\ j_1 \cdots j_n \end{pmatrix}\right)$ 为 $\det\left(A\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}\right)$ 余子式

②

例：

$$\begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

定理：

$$\det(A) = \sum_{i=1}^{n} a_{ki} A_{ki} \qquad 1 \le k \le n$$

证： 对 n 进行归纳.

· $n=2$ $\quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \left\{ \begin{array}{l} LHS := ad-bc \\ RHS := d\cdot a - c\cdot b \end{array} \right\} \Rightarrow \checkmark$

· 假设结论对 $n-1$ 阶行列式成立.

· D_{ij} := 删除 1,k 行, i,j 列的 $n-2$ 阶矩阵的行列式.

$D_{ij} = D_{ji}$.

· 归纳假设

$$\Rightarrow \left\{ \begin{array}{l} M_{1i} = \sum_{j=1}^{i-1} (-1)^{k-1+j} a_{kj} D_{ij} + \sum_{j=i+1}^{n} (-1)^{k-1+j-1} a_{kj} D_{ij} \\ M_{ki} = \sum_{j=1}^{i-1} (-1)^{j+1} a_{1j} D_{ij} + \sum_{j=i+1}^{n} (-1)^{j} a_{1j} D_{ij} \end{array} \right.$$

$$LHS = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} M_{1i}$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \left(\sum_{j=1}^{i-1} (-1)^{k-1+j} a_{kj} D_{ij} + \sum_{j=i+1}^{n} (-1)^{k-1+j-1} a_{kj} D_{ij} \right) \quad ③$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k} a_{1i} a_{kj} D_{ij} + \sum_{\bar{i}=1}^{n} \sum_{\bar{j}=\bar{i}+1}^{n} (-1)^{\bar{i}+\bar{j}+k-1} a_{1\bar{i}} a_{k\bar{j}} D_{\bar{i}\bar{j}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k} a_{1i} a_{kj} D_{ij} + \sum_{\bar{j}=1}^{n} \sum_{\bar{i}=1}^{\bar{j}-1} (-1)^{\bar{i}+\bar{j}+k-1} a_{1\bar{i}} a_{k\bar{j}} D_{\bar{i}\bar{j}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k} a_{1i} a_{kj} D_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k-1} a_{1j} a_{ki} D_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k} \left(a_{1i} a_{kj} - a_{1j} a_{ki} \right) D_{ij}$$

$$\text{RHS} = \sum_{i=1}^{n} (-1)^{k+i} a_{ki} M_{ki}$$

$$= \sum_{i=1}^{n} (-1)^{k+i} a_{ki} \left(\sum_{j=1}^{i-1} (-1)^{j+1} a_{1j} D_{ij} + \sum_{j=i+1}^{n} (-1)^{j} a_{1j} D_{ij} \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k-1} a_{1j} a_{ki} D_{ij} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (-1)^{i+j+k} a_{1j} a_{ki} D_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k-1} a_{1j} a_{ki} D_{ij} + \sum_{j=1}^{n} \sum_{i=1}^{j-1} (-1)^{i+j+k} a_{1j} a_{ki} D_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k-1} a_{1j} a_{ki} D_{ij} + \sum_{j=1}^{n} \sum_{i=1}^{j-1} (-1)^{i+j+k} a_{1j} a_{ki} D_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j+k} \left(a_{1i} a_{kj} - a_{1j} a_{ki} \right) D_{ij}$$

例：

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \iddots & \\ a_{n-1\,1} & a_{n-1\,2} & & \\ a_{n1} & & & \end{vmatrix} = (-1)^{\frac{(n+4)(n-1)}{2}} a_{n1} a_{n-1\,2} \cdots a_{1n}$$

定理：行列式性质

1) 交换 A 两行得矩阵 B，则 $\det(B) = -\det(A)$

2) A 的某行乘 λ 得矩阵 B，则 $\det(B) = \lambda \det(A)$

3) A 的某行是两向量之和，则 $\det(A)$ 可拆成两行列式之和.

4) A 的两行成比例，则 $\det(A)=0$

5) 将 A 的一行加上另一行的 λ 倍得 B，则 $\det(B)=\det(A)$

证：(1) $\det A = \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{p+q+i+j-1} (a_{pi}a_{qj} - a_{pj}a_{qi}) D_{ij}^{pq}$